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Sections of pyramids, perspectivities and conics

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An important part of classical geometry is the wide area of projective geometry. Here a brief and elementary introduction is given to this field based upon plane sections of pyramids in three-dimensional Euclidean space (see section 1). In section 2 conic sections are constructed as the perspective images of a circle. In the last section we construct conic sections given by five elements (points or tangents). Systematically we solve all of the six possible problems. This part is very much different from what can be found in the standard references on projective geometry but it gives good insight into the methods of this powerful tool.

1. Perspectives regarded as sections of pyramids

Consider a pyramid with top S and base ABC in the plane E (see figure 1). The pyramid is intersected by the plane E'. To get the section $A'B'C'$ in E', we introduce the parallel planes $F \parallel E$ through S and $F' \parallel E'$ through S. The dotted lines are the intersections of the sides of the pyramid with the planes E, E' and F'. Now the unknown figure $A'B'C'$ can be constructed simply by noting that corresponding intersecting lines in E' and F' are parallel. Figure 1 is the parallel-projection of the real situation in three-dimensional Euclidean space.

Now we remove all lines in figure 1 which suggest the embedding in space (but we keep the lines $r = E \cap E'$, $q = E \cap F'$ and $h = F \cap E'$). In this way we obtain in figure 2 Desargues' famous two-triangle theorem (see below).

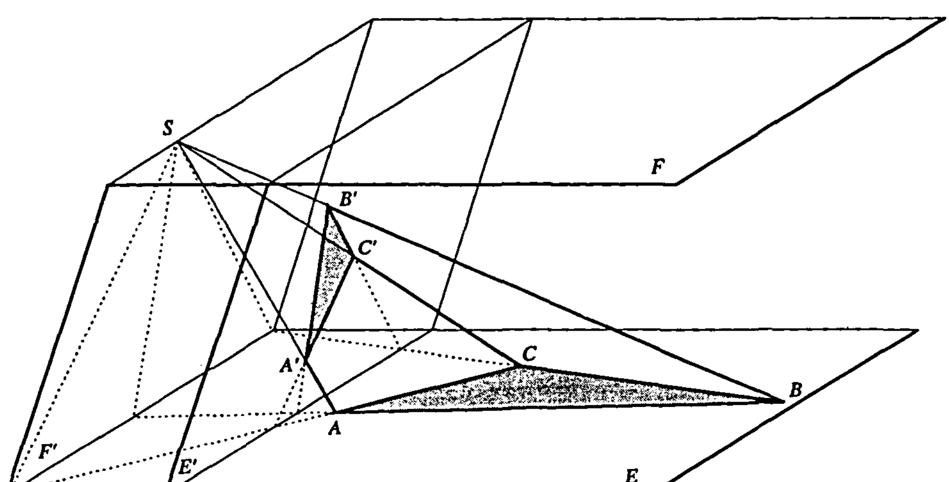

Figure 1. Section of a pyramid by a plane.

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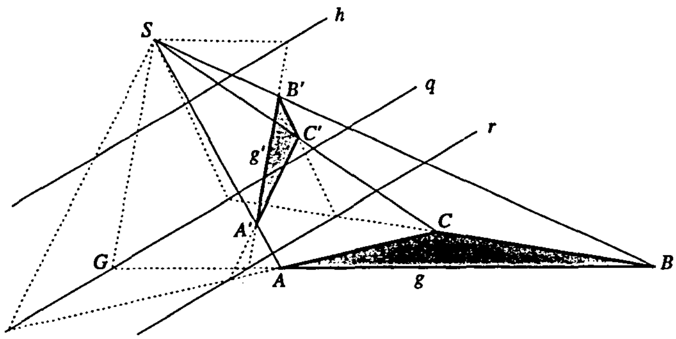

Figure 2. Perspectivity.

From Figure 2 we define the notion 'perspective transformation' and formulate the announced theorem:

Definition. A perspectivity (or perspective transformation) is given by a perspectivity centre S, a perspectivity axis r and an anti-axis q parallel to r. The image g' of a straight line $g \nparallel q$ is then defined as follows: g' is parallel to SG, $\{G\} = q \cap g$, and passes through $r \cap g$.

Convention. Image points and image lines respectively will be noted by a $'$ (prime).

Remark. Of course, the image A' of a point $A \notin q$ is defined as intersection-point of two lines through A.

The announced theorem is the following perspective 'closing-figure' (figure 2):

Theorem (Desargues). Copolar triangles (namely ABC and $A'B'C'$ in figure 2) in space or in a plane are coaxial and conversely. Here 'copolar' means that the lines AA', BB' and CC' intersect in S whereas 'coaxial' means that corresponding lines (as AB and $A'B'$) intersect on the straight line r.

Let us briefly summarize the most important properties of perspective transformations—we extract them directly from Figures 1 and 2:

(*a*) Point, image point and centre S are collinear.
(*b*) Line and image line intersect on the axis r.
(*c*) Line and parallel to its image line through S intersect on the anti-axis q.
(*d*) The images of points on q lie on the so called ideal line.
(*e*) Image line and parallel to the pre-image line through S intersection on the image h of the idea line (h is called hyper-axis of the perspectivity).
(*f*) The inverse mapping to the perspectivity with centre S, axis r, anti-axis q and hyper-axis h is the perspectivity with centre S, axis r, anti-axis h and hyper-axis q.

Remark. The notion of the ideal line gives rise to the projective closure of the Euclidean plane. But we do not go into this part of the theory.

2. The perspective image of the circle

In this section we will use some elementary knowledge of conic sections, such as the fact that the plane section of an oblique circular cone will be a conic. Thus, perspectivity seems to be an adequate tool to investigate conics, since those curves just appear as images of circles under perspective transformations.

Depending on the position of the circle K we have to consider three cases:

Case 1 $K \cap q = \varnothing$. Here the perspective image of K is an ellipse. In figure 3 we show how to construct the axis of this ellipse.

Description

- We choose $X, Y \in q$ such that $\measuredangle XSY = 90°$.
- The tangents from X and Y at K build a tangent-quadrangle around K with contact points A, B, C, D.
- The image of the tangent-quadrangle mentioned above is thus a rectangle, and hence $A'C'$ and B'D' are conjugate diameters of the image ellipse.
- By Rytz's construction we find the axis of the ellipse from its conjugate diameters.

Case 2 $K \cap q = \{P\}$. Here the perspective image of K is a parabola. In figures 4 and 5 we show the construction of its axis, the directrix and the focus.

Description

- SP is the direction of the axis of the image parabola.
- We choose $Q \in q$ such that $\measuredangle PSQ = 90°$.
- Let t be the tangent from Q at K and B the contact-point.
- Thus, B' is the vertex and t' the vertex-tangent of the image parabola. Hence we have also found the axis a of the parabola.

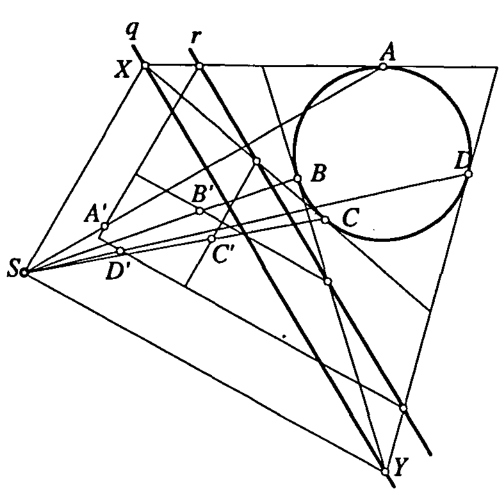

Figure 3. Construction of the axis of the image ellipse.

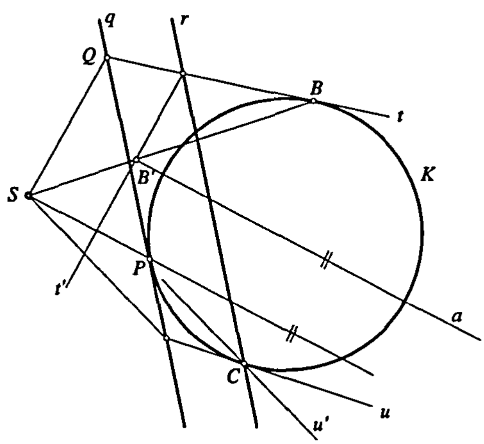

Figure 4. Construction of axis, vertex and a tangent of the image parabola.

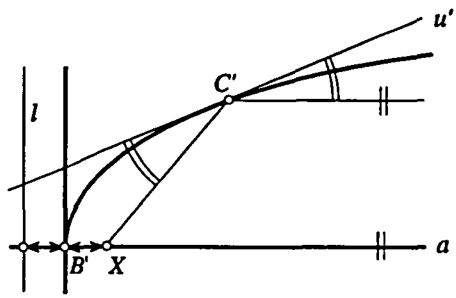

Figure 5. Construction of directrix l and focus X of a parabola from given axis a, vertex B' and tangent u' with contact-point C'.

- Let $C \in r \cap K$. Then the image u' of the tangent u at K through C may be constructed very easily. Since u' is a tangent to the parabola in $C' = C$, we get the focus and the directrix according to figure 5.

Case 3 $K \cap q = \{P, Q\}$. Here the perspective image of K is a hyperbola. In figure 6 we show the construction of its asymptotic lines and its vertices.

Description

- PS and QS are the asymptotic directions.
- Let w be the exterior angle bisector of PS and QS, and let $\{R\} = q \cap w$.
- Consider the tangents from R at K with contact-points A and B.
- Hence A' and B' are the vertices of the hyperbola, the midpoint M of $A'B'$ is the intersection point of the asymptotic lines.
- The construction of the focal points from M, the asymptotic lines and the vertices is then well-known.

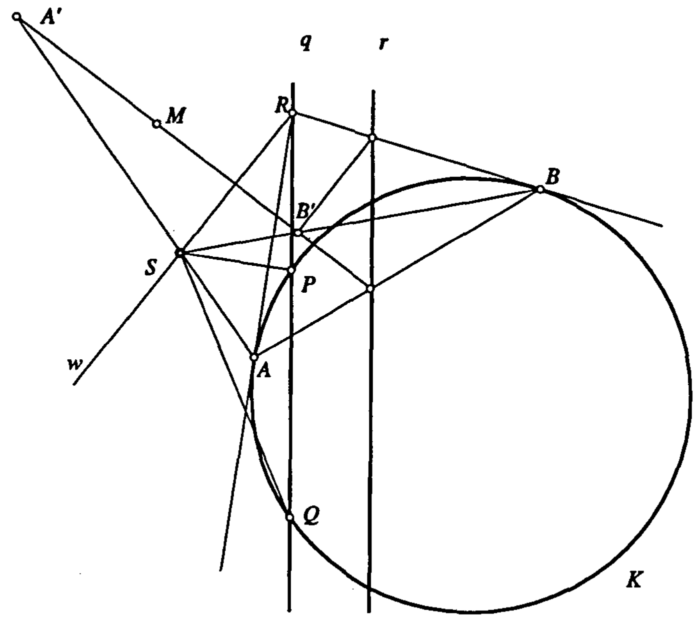

Figure 6. Construction of asymptotic lines and vertices on the image hyperbola.

3. Construction of conic sections from five given elements

A conic section is in general given by five elements (points P or tangents t) (we only consider general position of the elements). Hence there are six different conic section problems, namely:

Construct the conic section which is given by

(A1) P_1, P_2, P_3, P_4, P_5 (A3) P_1, P_2, P_3, t_1, t_2 (A5) P, t_1, t_2, t_3, t_4

(A2) P_1, P_2, P_3, P_4, t (A4) P_1, P_2, t_1, t_2, t_3 (A6) t_1, t_2, t_3, t_4, t_5

In any case the problem is solved, if it is possible to find a perspectivity and a circle K, such that the conic in question is just the image of K. The conic itself may then be found according to the constructions given in the previous section.

(A1) *Conic section through five points A', B', C', D', P'*

We look for a perspectivity such that the pre-image points $ABCD$ form a rectangle and P is a point on its circumcircle. The conic in question is then the perspective image of this circumcircle and the problem is solved. The perspectivity we are looking for may be found as follows: see figure 7.

Description

- Let $\{R\} = A'D' \cap B'C'$ and $\{T\} = A'B' \cap C'D'$.
- Choose $h = RT$ as hyper-axis.
- Let $\{U\} = A'P' \cap$ h and $\{V\} = P'C' \cap h$.
- The centre of the perspectivity S is chosen as intersection point of the Thales-circles over RT and UV. The axis $r \parallel h$ is arbitrary—it is now obvious that this perspectivity has the desired properties.

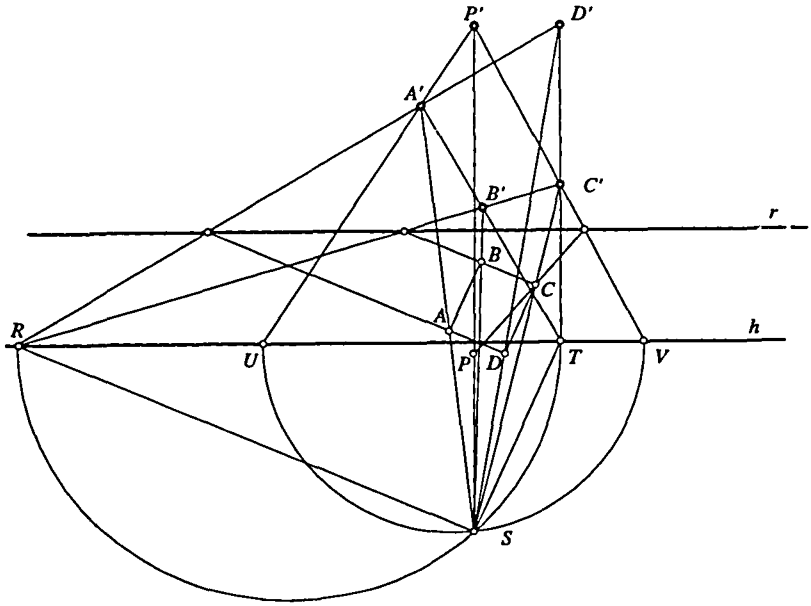

Figure 7.

(A2) *Conic section through four points A', B', C', D' and at a tangent t'*

This problem will be reduced to (A3): To do this consider the construction in figure 8. If t' is a tangent of the conic, then so is u'. We can see this if we look at the situation in the perspective image, where $ABCD$ is a square (see figure 9). (To see that there exists a perspectivity such that the pre-image of $A'B'C'D'$ is a square is an easy exercise.)

(A3) *Conic section through three points A', B', C' and at two tangents t_1', t_2'*

The problem is to find a perspectivity such that the pre-images A, B, C of A', B', C' lie on a circle K such that the pre-images t_1, t_2 of t_1', t_2' are tangents to K. In figure 10 we show how to find such a perspectivity.

Description

- Let K be a circle through the given points A' and B' (K is supposed to be the pre-image of the conic section we are looking for).
- As axis of the perspectivity we choose $r = A'B'$ (thus $A = A'$ and $B = B'$).

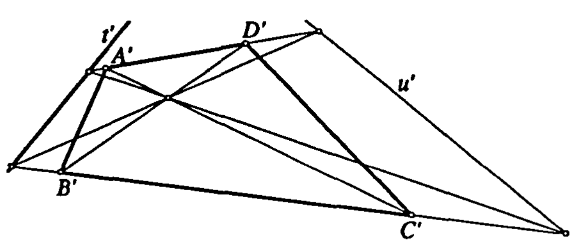

Figure 8.

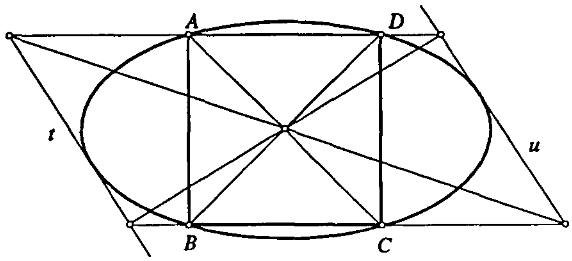

Figure 9

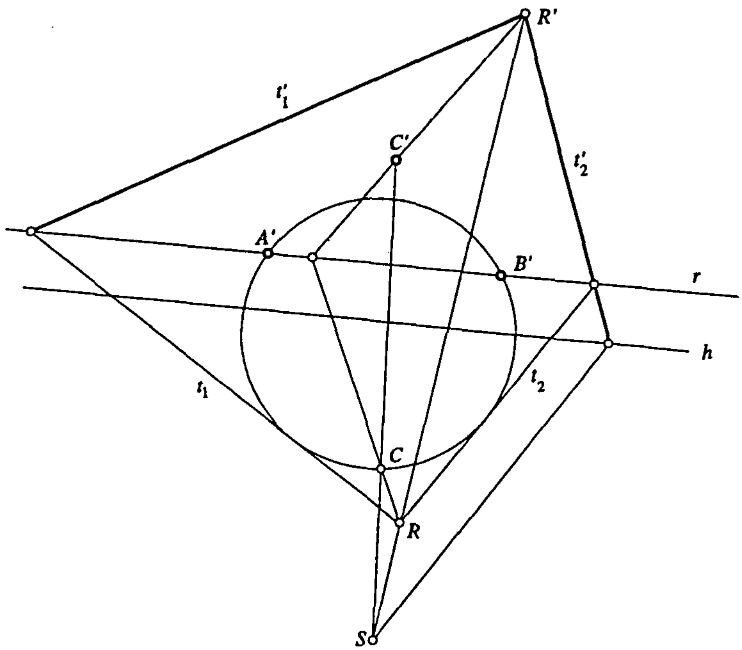

Figure 10.

- The images t_1 and t_2 of t'_1 and t'_2 are defined as tangents at K. The image of $\{R'\} = t'_1 \cap t'_2$ is $\{R\} = t_1 \cap t_2$.
- The pre-image C of C' must lie on K as well as the pre-image of the line $R'C'$ and it is hence already determined.
- The centre of the perspectivity S is determined as intersection point of the lines $R'R$ and $C'C$.
- The hyper-axis h can be found as follows: it is the parallel line to r through $t'_2 \cap g$, where g is the parallel to t_2 through S.

(A4) *Conic section through two points A', B' and at three tangents t'_1, t'_2, t'_3*

To solve this problem we have to find a perspectivity, such that the pre-images of t'_1, t'_2, t'_3 are tangents at a circle K such that the pre-images of A' and B' also lie on K. We will use the theorem we mentioned in section 1. The construction is given in figure 11.

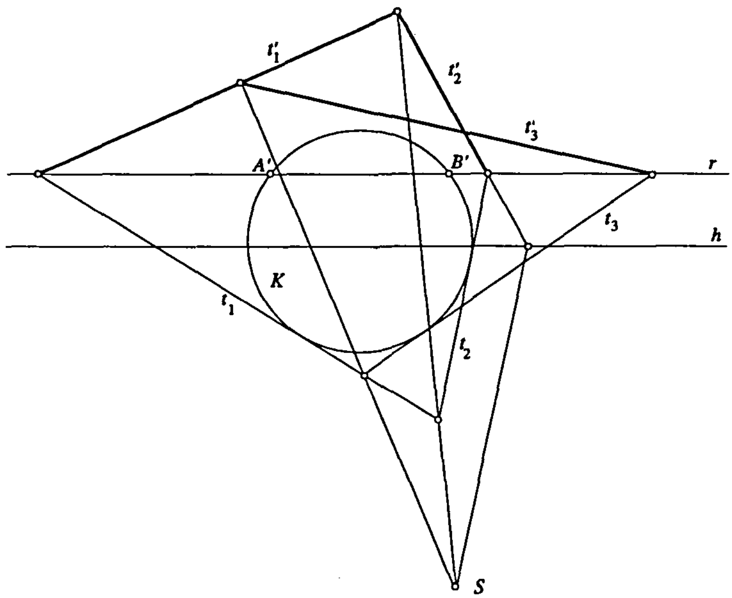

Figure 11

Description

- Let K be a circle through the given points A' and B' (K is supposed to be the pre-image of the conic we are looking for).
- As axis of the perspectivity we choose $r = A'B'$ (thus $A = A'$ and $B = B'$).
- The pre-image t_1, t_2, t_3 of the triangle built from t_1', t_2', t_3' is then chosen such that K is an excircle.
- Since the sides of the triangles built from t_1', t_2', t_3' and t_1, t_2, t_3 intersect on r, the lines joining corresponding edges intersect in a point S which we choose as centre of the perspectivity.
- The hyper-axis h is now constructed as in the previous construction.

(A5) *Conic section through a point P' and at four tangents t_1', t_2', t_3', t_4'*

This problem will be reduced to problem (A4). To do this we consider figure 12. If P' is a point on the conic, then so is Q'. This can be seen, if we look at the situation in the perspective image when t_1, t_2, t_3, t_4 is a square (see figure 13).

(A6) *Conic section at five tangents t_1', t_2', t_3', t_4', g'*

To solve problem (A6) we first make the following observation, which we develop in figure 14. If K is the incircle of an equilateral parallelogram with sides t_1, t_2, t_3, t_4 and g is a tangent at K which intersects t_1 in P and t_3 in Q, then $\measuredangle PMQ = 90°$, where M denotes the centre of K.

To solve problem (A6) we look for a perspectivity having the property that the pre-image of t_1', t_2', t_3', t_4' is an equilateral parallelogram such that the pre-image of g' is a tangent at the incircle of the equilateral parallelogram. The construction is given in figure 15.

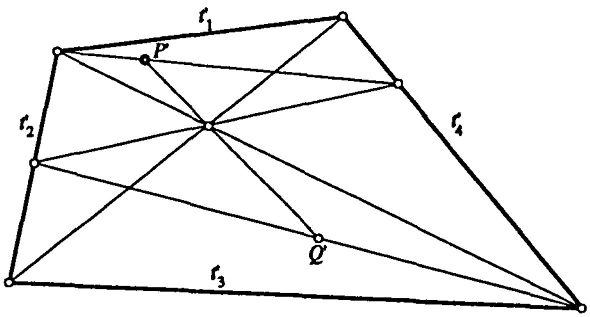

Figure 12.

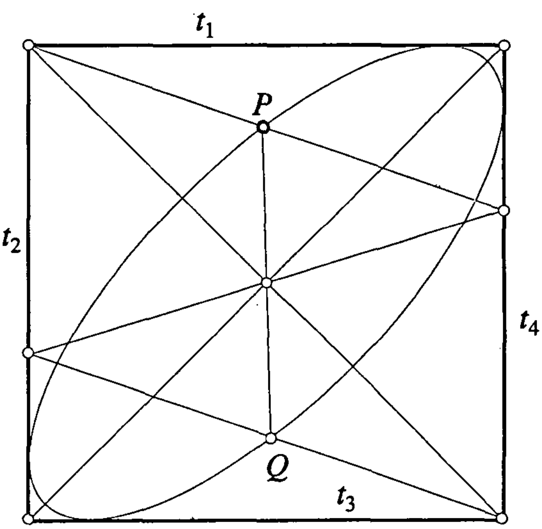

Figure 13.

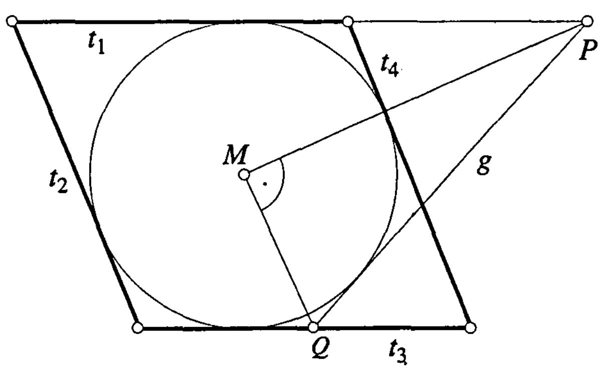

Figure 14.

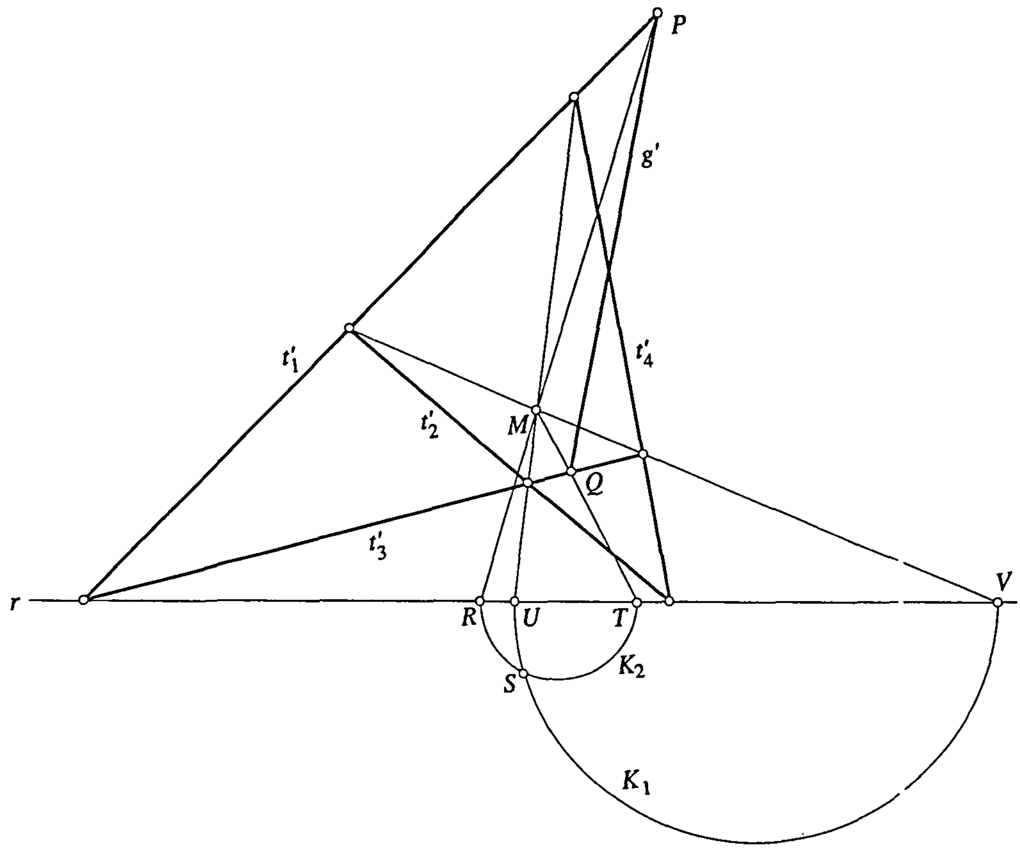

Figure 15.

Description

- As axis r of the perspectivity we choose the line through $t'_1 \cap t'_3$ and $t'_2 \cap t'_4$.
- Let $\{P\} = t'_1 \cap g'$ and $\{Q\} = t'_3 \cap g'$.
- Let U, V be the intersection points of the diagonals in the quadrangle with sides t'_1, t'_2, t'_3, t'_4 with r. We denote the intersection point of the diagonals by M.
- Let K_1 be the Thales-circle over UV.
- Let R be the intersection of r with MP and T the intersection of r with MQ.
- Let K_2 be the Thales-circle over RT, then we choose the centre S of the perspectivity as intersection point of K_1 and K_2. The hyper-axis may now be found as in the previous construction. The pre-image of t'_1, t'_2, t'_3, t'_4 is then in fact an equilateral parallelogram and the pre-image of g' is a tangent to its incircle.

Remark. It is not the idea to give in the Reference section a complete survey on the literature of projective geometry. We only give three references which are also readable for non-specialists and which also give hints for more advanced books.

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